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VECTOR POWER IN ALTERNATING-CURRENT CIRCUITS

BY A. E. KENNELLY

It has long been known that in any simple alternating-current circuit, the current and voltage may be conveniently regarded as rotatable vector quantities.¹ It is also known that the power in such circuits is not to be regarded as the vector product of the rotating vector voltage and rotating vector current.² It does not seem to have been pointed out, however, that, under certain restrictions, it is proper to regard the power in an alternating-current circuit as a non-rotating vector quantity. Moreover, it does not appear to be generally known, although the fact has not escaped notice, that the imaginary component of vector power, or so-called "wattless power" is, in a restricted sense, just as much power, and just as "wattful" as the real component.³

The objects of this paper are:

1. To indicate the limitations under which power in an alternating-current circuit may be treated as a stationary vector;
2. To extend the technology of vector alternating-current quantities;
3. To combat the use of the terms "wattless power" and

1. J. A. Fleming, "Notes on Alternate Currents", "The Electrician", Nov. 18, 1887, Vol. 20, page 28.

2. Symbolic Representation of General Alternating Waves and of Double Frequency Vector Products, by Dr. C. P. Steinmetz, TRANSACTIONS A.I.E.E., Vol. 16, pp. 269-296. June, 1899.

The use of Complex Quantities in Alternating Currents, by Geo. W. Patterson, Physical Review, Vol. 26, No. 3, Mar. 1908, pp. 266-271.

3. The Improvement of Power Factor in Alternating-Current Systems, by Miles Walker, Journal of the Inst. of El. Engrs., Vol. 42, pp. 599-625, Jan., 1909.

“ wattless current ”, offering more logical terms as substitutes, and

4. To offer a plea for the standardization of the direction of phase rotation in the vectors used in alternating-current theory.

Preliminary Definitions. The vectors, or directed magnitudes, employed in alternating-current technology, with rare exceptions that do not come within the scope of this discussion, are all confined to a single plane of reference. That is to say, they relate to two dimensions of space, as distinguished from the vectors of three-dimensional geometry. This limitation may be expressed by saying that the vectors of alternating-current technology are *plane vectors*. A plane vector may be defined as a quantity having both a direction and a magnitude, but confined to one plane of reference. In what follows, we may for brevity conveniently assume that the term “ vector ” is an abbreviation for the more strictly logical term “ plane vector ”.

Subdivision of Vectors. There are three classes of vector used in dealing with alternating-current circuits, namely:

1. Vectors that are capable of rotation in their reference plane about a fixed point, and whose projections on a reference axis, or whose intercepts with polar curves, measure the instantaneous values of the quantities represented by the vectors. That is, the rotating vectors may be either *projected* or *intercepted*. These vectors may be called *rotative vectors*.

2. Vectors that are not capable of rotation in their reference plane for any purpose of projective or interceptive representation. These may be called *non-rotative vectors*.

3. Rotative-vectors that for special purposes are arrested, or treated as though non-rotative. These may be called *stationary vectors*. Stationary vectors are rotative, but non-rotating.

The above classification and nomenclature may be revealed more clearly by the following table:

TABLE I
Nomenclature and algebraic classification of alternating-current vectors.

Rotative ($l / \omega t$)	$\left\{ \begin{array}{l} \dots\dots\dots \text{Rotating } (l / \omega t) \\ \text{Stationary } (l / \omega_* T) \\ (T = \text{constant}) \end{array} \right\}$	Non-rotating (l / θ)
Non-rotative (l / θ)		

Example of a Rotative Vector. As an example of class (1), let us consider the vector OE , Fig. 1, rotating in the plane OXY , about the origin O , with a uniform angular velocity of ω radians per second. Then the length OE may represent to an assigned

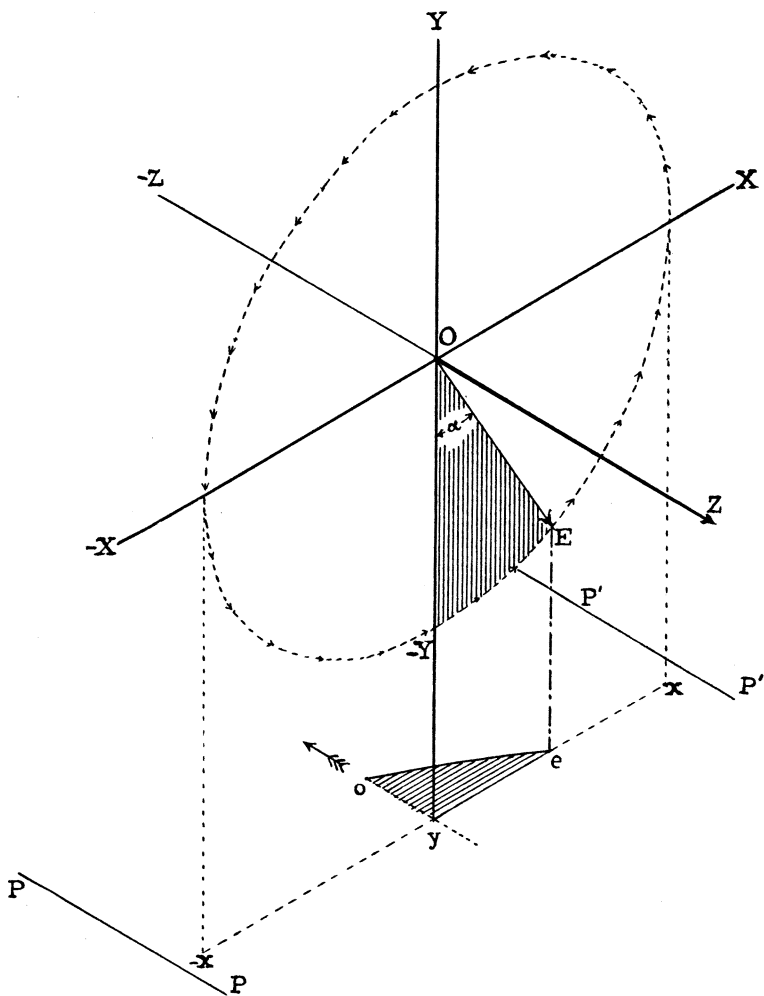


FIG. 1.—Rotative vector and sinusoidal projection. Isometric projection. XOY plane of rotation, $PP'P'P'$ plane of projection.

scale of volts per cm., the maximum cyclic value of a certain sinusoidal e.m.f., say the e.m.f. generated harmonically in the secondary winding of a particular transformer. The direction of rotation may be positive, as indicated by the orbital arrows.

If the frequency of the voltage in the transformer is n cycles per second; then we know that the angular velocity of rotation must be $\omega = 2\pi n$ radians per second. Moreover, the vector OE must, of course, coincide with OX when the generated voltage attains its maximum positive value.

The circular orbital motion of the end E of the rotation vector will then correspond to the vector equation:

$$OE = E_0 \varepsilon^{j\omega t} \quad \text{volt-scale cm. } \angle \quad (1)$$

where t is the time, in seconds, from the start at a particular axis such as OX , ε is 2.71828 . . . , the base of Napierian logarithms, $j = \sqrt{-1}$, and E_0 is the maximum cyclic value of the e.m.f. to a voltage scale of length. Equation (1) is sometimes written:

$$OE = E_0 \text{ cis } (\omega t) \quad \text{volt-scale cm. } \angle \quad (2)$$

We shall represent either of the above expressions by the briefer and more convenient notation:

$$OE = E_0 \angle \omega t \quad \text{volt-scale cm. } \angle \quad (3)$$

where the quantity ωt within the angle sign $\angle$ means that the angular distance of the radius vector OE from the initial reference axis is ωt radians, or degrees, according to the unit of angle adopted.

As is well known, the orthogonal projection of the radius vector OE upon the plane of reference $PP, P'P'$, which is parallel to the plane $OX Y$, performs a simple harmonic motion. If the reference axis of starting, at time $t = 0$, is $O - Y$, then at any instant, t seconds thereafter, the distance ye , or projection of OE , will be:

$$\overline{ye} = E_0 \sin \omega t = E_0 \sin \alpha \quad \text{volt-scale cm. } (4)$$

which will correspond to the e.m.f. generated at that instant in the transformer.

Again, if the reference axis of starting, at time $t = 0$, is OX , then at any time t :

$$\overline{ye} = E_0 \cos \omega t = E_0 \cos \alpha \quad \text{volt-scale cm. } (5)$$

If the plane of projection $PP P' P'$ be moved parallel to itself in the direction shown by the arrow at o ; or, if with the plane

of projection at rest, the coördinate system of axes moves in the direction OZ , with uniform linear velocity, then, as is well known, the projecting point will trace out a sinusoid oe having amplitude ordinates ox , and time abscissas oy .

RELATION OF PHASE TO THE DIRECTION OF ROTATION

Convention No. 1. If we employ two co-frequent rotative vectors, such as OE and OI , Fig. 2, representing say an im-

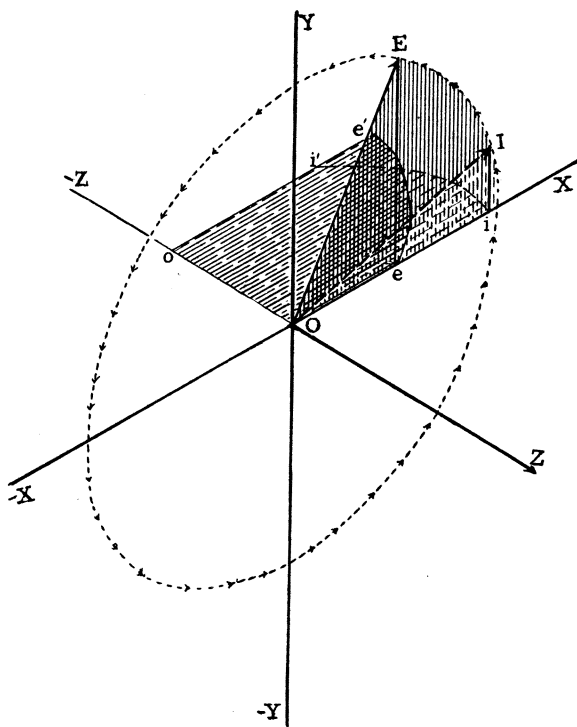


FIG. 2.—Pair of rotative vectors of the same frequency and their cosinusoidal projections. Representation direct. Current lagging. Isometric projection. XOY plane of rotation. XOZ plane of projection.

pressed e.m.f., to a certain volt-scale, in a simple alternating-current circuit, and the current strength thereby produced in the same circuit, to a certain ampere-scale; then, if there is inductive reactance in the circuit, we know that the current will lag behind the impressed e.m.f. This means that the orthogonal projection of OI on the axis OX of reference, must reach its maximum after OE has passed its maximum projection on that

axis. Consequently, with the plane $X O Z$, fixed in space, and with the vectors $O E$, $O I$, rotating in the positive direction, as shown by the orbital arrows, $O I$ must make a negative angle with $O E$; or, $O E$ must make a positive angle with $O I$; so that the angle $E O I$ is trigonometrically *negative*, as shown in Fig. 2. If the vector $O E$ starts from the position $O X$, at time $t = 0$, the orthogonal projection of $O I$ $/\omega t - \theta$ proceeds to execute on the axis $O X$ the cosinusoid:

$$\overline{O i} = I_0 \cos (\omega t - \theta) \quad \text{ampere-scale cm. (6)}$$

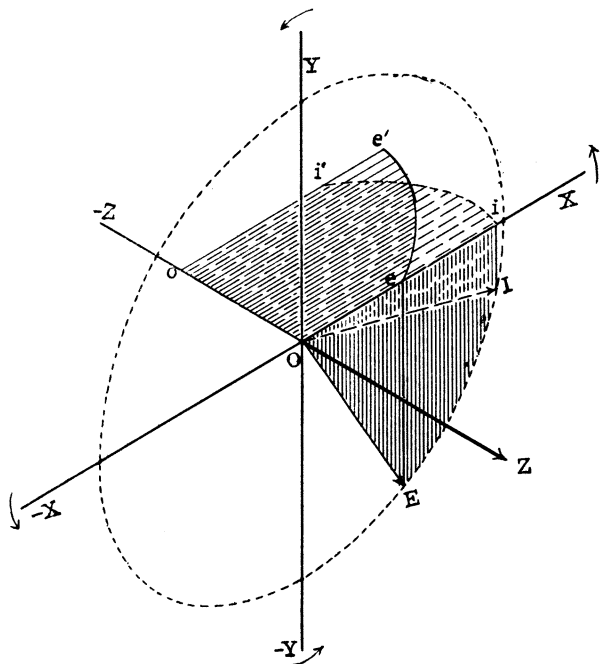


FIG. 3.—Pair of rotative vectors of the same frequency and their cosinusoidal projections. Representation inverse. Current lagging. Isometric projection. $X O Y$ plane of rotation. $X O Z$ plane of rotation.

If the origin O and system of axes advance uniformly with respect to the stationary plane of projection $X O Z$ in the direction $O Z$; or, what is equivalent, if the plane of projection moves in the direction $O Z$ past a fixed origin O , the cosinusoids $e' e$ and $i' i$ will be traced as curves with amplitude ordinates, and times as abscissas. This convention was adopted in alternating-current technology by Fleming in 1887.¹

1. Fleming, loc. cit.

Convention No. 2. If, however, we reverse the direction of rotation of the vectors OE , OI in their orbit; *i.e.*, if we adopt negative rotation, we shall require that the leading vector make a negative angle with the lagging vector, a condition opposite to that above defined.² Or, keeping the direction of rotation positive, we may assume that the vectors are fixed in their plane; but that the plane of projection rotates positively at the uniform angular velocity ω . Thus in Fig. 3, the two vectors OE and OI may respectively represent, as before, the impressed e.m.f. and the current in an inductively reactive circuit. If these two vectors remain fixed; but the axis $-XOX$, and with it the plane of projection ZOX , rotates in the positive direction, of the curved arrows, about the OZ axis, the e.m.f. vector OE must make a negative angle with the lagging current vector OI ; or the pair will take relatively opposite positions to those they had in Fig. 2, and the angle EOI will be *positive*.

Convention No. 3. Some writers, instead of employing rotative vectors to be projected orthogonally upon an axis of reference, prefer to represent simple harmonic motion by the device of an *intercepting circle*.³ Thus, in Fig. 4, the heavy circle $OGEF$, in the plane XOY , may be regarded as stationary in space, and the axis OR , sometimes called a "time-axis," rotates positively in this plane about the origin O , as shown by the curved arrows, with uniform angular velocity ω radians per second, or n revolutions per second, commencing at the position $O-Y$, when $t = 0$. As the axis OR advances, it becomes intercepted by the circle $OG E$, and forms to that circle a chord of increasing length, until it reaches the position occupied in the Figure by $-XOX$, when the chord will have become a diameter, and the length of the moving axis intercepted by the fixed circle will be a maximum in the positive direction. As the rotation of OR about O continues, the length intercepted by the circle will diminish, until it will be zero in the position OY . Continuing the rotation, we may adopt either of two equivalent conventions, between which writers are divided. We may either use a second circle, shown in dotted lines at $Ogef$, equal and opposite to the first, and consider this to be a negative circle, such that all intercepts upon

2. Gisbert Kapp, "Alternate-Current Machinery", Fig. 7, Proc. Civil Engineers, London, 1889, reprinted in Van Nostrand Science Series, New York, 1889, p. 27.

3. Kapp, loc. cit., Fig. 8.

C. P. Steinmetz, "Alternating-Current Phenomena", 1897.

OR shall be considered negative, intercepts on Or being ignored; or, we may dispense with the second circle, and allow intercepts on Or to count as negative intercepts, during the second half of the revolution. In either case, the length of the intercept on the moving axis will follow a simple harmonic law, according to the expression $OE \sin \omega t$ units of length, OE being the diameter of the intercepting circle.

If the rotating axis starts, at time $t = 0$, from the position $-XOX$ in Fig. 4, the length intercepted by the fixed circle will likewise follow a simple harmonic law according to the expression $OE \cos \omega t$.

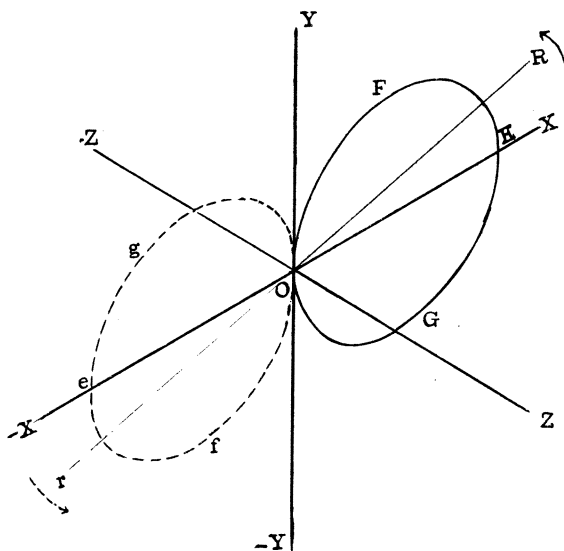


FIG. 4.—Uniformly rotating vector and fixed intercepting circle, with simple harmonic intercept. Isometric projection. XOY plane of rotation.

In order to represent the relation between a simple harmonic current in an inductively reactive circuit under a simple harmonic e.m.f. by convention 3, two intercepting circles are required, as in Fig. 5. Here the negative counterparts are omitted, and negative intercepts are assumed with each circle. As shown in the figure, it is necessary, with the positive direction of rotation of the moving axis OR , to have the diameter of the circle OI make a positive angle with the diameter of the circle OE , in order that the intercept on the current circle shall reach its maximum later than the intercept on the voltage circle. A lagging current, therefore, requires a *positive* angle EOI .

In the practical use of convention 3, the circles are commonly omitted, for convenience, and are merely represented by their diameters OE and OI , which become respectively the stationary-vector e.m.f. and current of the diagram.

As regards the use or disuse of the negative circle, as in Fig. 4, it is simpler to dispense with it, and to use negative intercepts on OY , except when the positive and negative waves are not symmetrical. In that case, the retention of the negative loop simplifies the diagram, since it avoids superposition of loops, and ambiguity of paths.

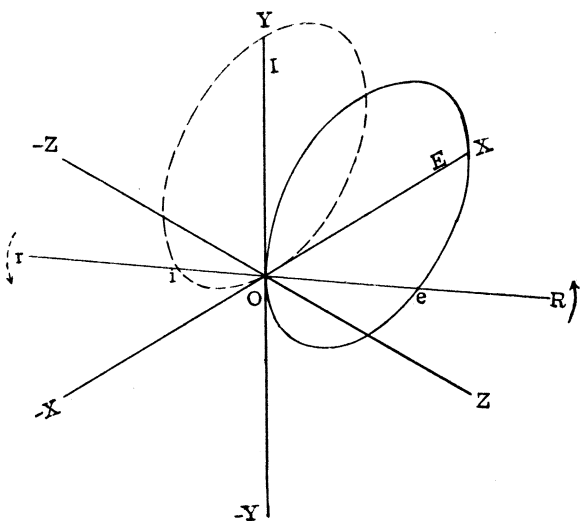


FIG. 5.—Pair of intercepting circles representing a simple harmonic e.m.f. and a simple harmonic current lagging 90 deg. behind the same. Isometric projection. XOY plane of rotation. Representation inverse.

Convention No. 4. If we assume that the axes of Fig. 5 are fixed in space, but that the intercepting circles OE and OI rotate together in the positive direction about the center O , measuring off from moment to moment intercepts on some axis, say OX , equal to the respective instantaneous voltage and current; then in order to represent an e.m.f. and a lagging current, it will be necessary for the diameter of OE to make a positive angle IOE with the diameter of OI , or the angle EOI will be *negative*, the opposite condition to that in Fig. 5. According

then to convention 4, Fig. 5 would represent a current $O I$ leading the e.m.f. $O E$ by 90 degrees.¹

Convention No. 4 does not seem to be in use, but is introduced here in order to complete the classification symmetrically, see Table II.

TABLE II
Conventions Employed in Rotative - Vector Diagrams

Convention		Direction of rotation	Plane of projection	Vectors	Intercepting circles	Vector	Angle EOI for an inductive circuit	User	Date of use	Expression of inductive impedance	Representation
Type	No.										
Orthogonal projection or clock diagram	1	$\left\{ \begin{array}{l} + \\ - \end{array} \right.$	Fixed	Rotating			$- \left\{ \begin{array}{l} \text{Fleming} \\ \text{Blakesley} \end{array} \right.$	1887	1889	$r + jx$	Direct
			Rotating	Fixed			$- \dots$			$r + jx$	
	2	$\left\{ \begin{array}{l} + \\ - \end{array} \right.$	Rotating	Fixed			$+ \dots$			$r - jx$	Inverse
			Fixed	Rotating			$+ \text{Kapp}$	1889		$r - jx$	
Polar coordinates or intercepting circles	3	$\left\{ \begin{array}{l} + \\ - \end{array} \right.$		Fixed	Rotating	$+ \left\{ \begin{array}{l} \text{Kapp} \\ \text{Steinmetz} \end{array} \right.$	1889	1893		$r - jx$	Inverse
				Rotating	Fixed	$+ \dots$				$r - jx$	
	4	$\left\{ \begin{array}{l} + \\ - \end{array} \right.$		Rotating	Fixed	$- \dots$				$r + jx$	Direct
				Fixed	Rotating	$- \dots$				$r + jx$	

REPRESENTATION OF COMPLEX HARMONIC QUANTITIES

It has been claimed that a complex harmonic quantity is only capable of being represented as a closed curve to polar coordinates by the method of the intercepting curve, and that the projecting curve, or "clock-diagram", cannot be used in

1. The above four conventions by no means exhaust the possibilities of rotative-vector representation. For example, as a subtype of interceptive or polar representation, we might assume the two circles OE, OI of Fig. 5 to have their diameters on one and the same line, instead of being angularly displaced. Two angularly rotating displaced vectors could then be employed for e.m.f. and current intercepts respectively, instead of the single rotating vector OR . With such an arrangement, a lagging vector current OI would make a negative angle with the e.m.f. vector OE , or would reverse the relations of Fig. 5. That is, it would produce direct representation. Since, however, this method does not seem to have been used, it is omitted from the Classified Table of Conventions.

such cases.¹ It is true that the simple projecting-circle or clock-diagram, with uniform angular velocity of the radius vector, cannot represent a non-sinusoidal or complex harmonic wave; just as it is true that the simple intercepting circle, with uniform angular velocity of the radius vector, cannot represent a complex harmonic wave. But in the same manner that a change in the

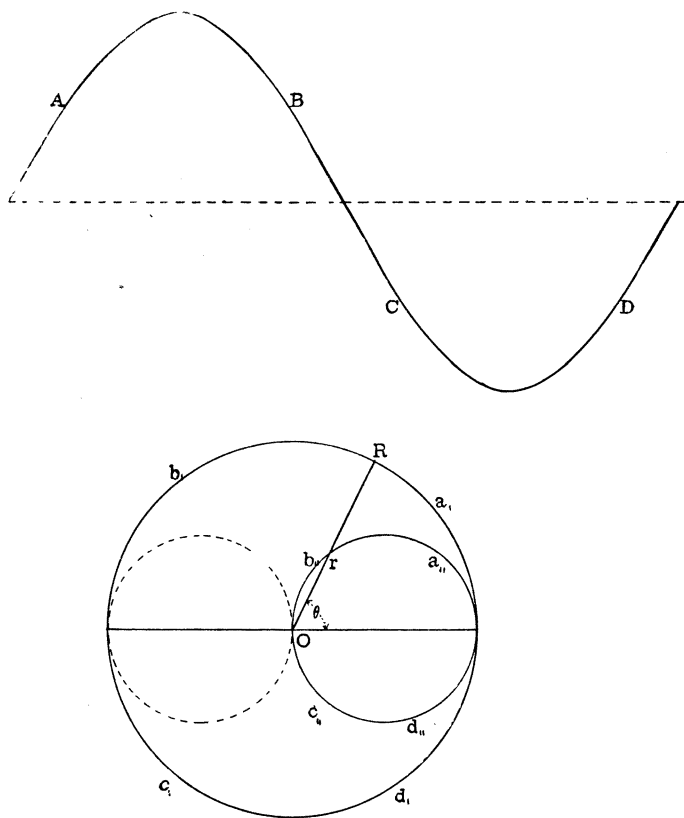


FIG. 6.—Sinusoidal wave and its projecting circle and its intercepting circle.

form of intercepting curve will permit of any complex wave being presented in polar coördinates, so a corresponding change in the form of the projecting curve will permit of the same result. Thus in Fig. 6, the sinusoid *A B C D*, drawn to rectangular coördinates, may be represented either by the projecting circle

1. *Alternating-Current Phenomena*, by C. P. Steinmetz, 4th Edition, 1908, p. 44.

a, b, c, d , or by the intercepting circle $a_{\mu} b_{\mu} c_{\mu} d_{\mu}$, with or without its dotted neighbor. Similarly, in Fig. 7, the triangular wave $A B C D$, drawn to rectangular co-ordinates, may be represented either by the projecting curve a, b, c, d , or by the intercepting curve $a_{\mu} b_{\mu} c_{\mu} d_{\mu}$, with or without its dotted neighbor. The relation between corresponding radii on the polar curves is always:

$$\overline{O}r = \overline{O}R \cos \theta \quad \text{cm. (7)}$$

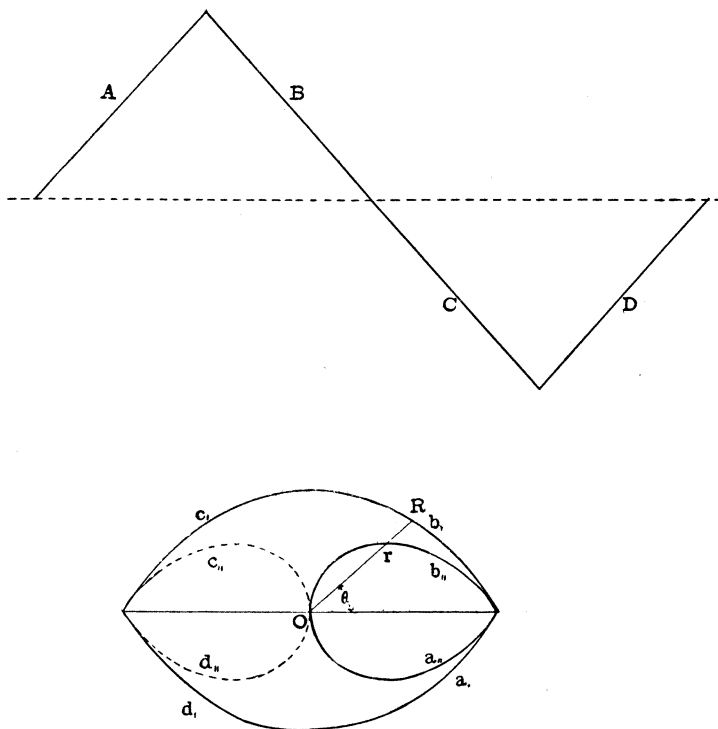


FIG. 7.—Triangular wave, its projecting curve, and its intercepting curve.

where $\overline{O}r$ is the radius vector (Figs. 6 and 7) of the intersecting curve, at the angle θ with the initial line of reference, and $\overline{O}R$ is the corresponding radius vector of the projecting curve. Consequently, having given either the polar projecting curve, or the polar intercepting curve of any complex harmonic alternating-current wave, the other can be immediately deduced.

It will be evident from the above considerations that with a pair of rotative vectors (Fig. 8) in the plane $X O Y$, one $O E$

representing an impressed e.m.f., and the other $O I$ the resulting current, in a simple alternating-current circuit, the question as to whether $O I$ is to be interpreted as a leading or a lagging current does not depend upon the use of projecting as against intercepting curves. It may be either a leading or a lagging current with either the "clock" diagram (Fig. 1) or the "spectacles" diagram (Fig. 4). It depends entirely upon the convention employed. If, with a projecting curve, the two vectors of Fig. 8 rotate (in the positive direction) as in convention No. 1, then $O I$ represents a lagging current. If on the contrary, the vectors are to be considered as stationary, and the axis of refer-

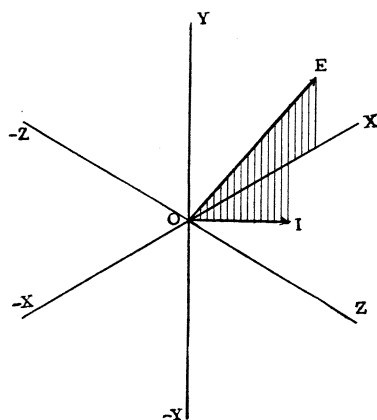


FIG. 8.—Vector e.m.f. and current in simple alternating-current circuit. The current is either a leading or a lagging current according to which convention used in vector representation. Isometric projection. $X O Y$ plane of rotation.

ence rotates positively with respect to them, as in convention No. 2, then $O I$ represents a leading current. Again, if with a fixed intersecting axis $O X$, a pair of circles located with their rotating diameters on $O E$ and $O I$, the vectors of Fig. 8 rotate in the positive direction, as in convention No. 4, $O I$ represents a lagging current. Finally, if the vectors $O E$ and $O I$ are fixed, and represent the diameters of fixed intersecting circles, with respect to which an axis of reference rotates positively, as in convention No. 3, $O I$ represents a leading current.

Aside from mental habit and psychological inertia, any one of these conventions appears to be as good as another. If the positive direction of rotation is understood in all cases, each con-

vention appears to be logical and systematic. It cannot be maintained that one or more are right and the rest are wrong. The question is essentially one of arbitrary convention, not of demonstration. Nevertheless, it is very important that the matter should be settled definitely by general agreement.¹ Much ambiguity results from the present dissension; because it is often difficult to ascertain, when opening a text-book in order to consult it, which method of representation the author follows. Imagine the confusion which would ensue in the worlds of pure and applied mathematics, if it were left to the choice of each writer to select which direction of the axis — $X O X$ Fig. 8, say, should be positive; so that some books should be written on the assumption of $O X$, Fig. 8, being plus, and others on the basis of $O X$ being minus. Or again, suppose it were left open to individual selection, which direction of rotation in a plane, clockwise, or counter-clockwise, should be taken as positive. These arbitrary selections have long been fixed by universal agreement among mathematicians. Yet this is the kind of dissension which exists to-day in the world of vector alternating-current technology.¹

Moreover, it is not enough that the decision should be made and accepted nationally. The only satisfactory decision must be made and accepted internationally.

DIRECT AND INVERSE REPRESENTATION

In what follows, this paper will conform to what seems the majority of opinion on this matter, and $O I$ in Figs. 8 and 10, will be regarded as a lagging current with respect to the e.m.f. $O E$. This means adhering to conventions 1 and 4, and by preference to convention 1. This method of representation which, in the direction of positive rotation, makes a *leading* current *lead*, and a *lagging* current *lag*, with respect to its e.m.f., will be called, for the purposes of distinction, *direct representation*, and the opposite method, involving either convention 2 or convention 3, will be called *inverse representation*.

Tables III and IV contain lists, which are by no means exhaustive, of publications using direct and inverse representations

1. The urgent need for the standardization of alternating-current vector rotation has been pointed out by various writers, both in this country and in Europe. See W. S. Franklin: A discussion of some points in Alternating-Current Theory. TRANSACTIONS A.I.E.E., May 1903, Vol. 21, pp. 589-601, and Carl Richter, Alternating-current Diagrams, Elek. u. Maschinenbau, July 12, 1908, pp. 608, 609.

respectively. No attempt has been made to discover more publications using one method than the other, and in the search that led to the formulation of these lists, no publications which contained vector diagrams were discarded, except such as made it difficult to decide which method was followed. The fact that 42 publications appear in the list of direct representation and 24 in the list of inverse representation, indicates a distinct preponderance in favor of direct representation. The dissension is not confined to any single country, or group of countries, and it dates as far back as 1889 at least.

TABLE III

PUBLICATIONS EMPLOYING DIRECT REPRESENTATION OF ALTERNATING-CURRENT VECTORS

Name of publication	Author	Publisher	Date
1. Conductors for Electrical Distribution.	F. A. C. Perrine	D. Van Nostrand, N. Y.	1907
2. Electrical Engineer's Pocket Book	H. A. Foster	"	1908
3. Alternating-Current Machines.	Sheldon, Mason and Hausmann	"	1909
4. Standard Handbook for Electrical Engineers		McGraw Pub. Co., N. Y.	1908
5. Munroe & Jamieson's Pocket Book		Ch. Griffin & Co., London	1908
6. Direct and Alternating Current Testing	F. Bedell	D. Van Nostrand, N. Y.	1909
7. Elements of Electrical Engineering	Franklin & Esty	MacMillan Co.	1909
8. Electric Waves	W. S. Franklin	"	1909
9. Standard Polyphase Systems	M. A. Oudin	D. Van Nostrand	1909
10. The A. C. Transformer	F. G. Baum	McGraw Pub. Co.	1903
11. Electric Power Transmission	L. Bell	"	1907
12. Problems in Electrical Engineering	W. V. Lyon	"	1908
13. Electrical Engineering Leaflets	Houston & Kennelly	The Elec. Engineer	1897
14. A Laboratory Manual of Physics and App. El.	E. L. Nichols	MacMillan Co.	1894
15. Alternating Currents	Bedell & Crehore	W. J. Johnston Co.	1893
16. Alternating Currents	D. C. & J. P. Jackson	MacMillan Co.	1896
17. Electrical Transmission of Energy	A. V. Abbott	D. Van Nostrand	1904
18. Alternating-Current Motors	A. S. McAllister	McGraw Pub. Co.	1907
19. The Elements of Alternating Currents	Franklin & Williamson	MacMillan Co.	1901
20. Alternating-Current Machinery	W. Esty	Am. Sch. Corresp.	1909
21. Electrical Measurements	Carhart & Patterson	Allyn & Bacon	1895
22. A Text Book of Electrical Machinery	Ryan, Norris and Hoxie	John Wiley	1903
23. The Principles of A. C. Working	A. Hay	Biggs & Co.	1897
24. Telephone Lines and their Properties	W. J. Hopkins	Longmans Green	1894
25. Experimental Electrical Engineering	V. Karapetoff	John Wiley	1908
26. Electrical Problems	Hooper & Wells	Ginn & Co.	1902
27. The Dynamo	Hawkins & Wallis	MacMillan Co.	1909
28. Electricity and Magnetism	F. E. Nipher	J. L. Boland Co.	1895

Name of publication	Author	Publisher	Date
29. Alternating Currents	C. G. Lamb	Ed. Arnold, London	1906
30. Leçons d'Electrotechnique v Générale	P. Janet	Gauthier-Villars	1908
31. Pratique Industrielle des Cour- ants Alt:	G. Chevrier	Carré & Naud	1900
32. Recherche Élémentaire des Relations:	O. de Bast	Léon de Thier	1899
33. Distribution de l'Energie par Courants Polyphases	J. Rodet	Gauthier-Villars	1903
34. La Technique de la Houille Blanche	E. Pacoret	Dunod et Pinat	1908
35. Moteurs Synchrones	A. Blondel	Gauthier-Villars	—
36. Die Wissenschaftlichen Grund- lagen der Elek:	G. Benischke	J. Springer	1907
37. Fernleitung von Wechselströmen	G. Roessler	J. Springer	1905
38. Handbuch der Elektrischen Beleuchtung	Herzog & Feldmann	"	1907
39. Messungen an Elektrischen Maschinen	R. Krause	"	1907
40. Ein-und Mehrphasen W: Erzeuger	F. Niethammer	Hirzel	1900
41. Der Drehstrom	J. Krämer	H. Costenoble	1896
42. La Tecnica delle Correnti Alter- nate	G. Sartori	Ulrico Hoepli	1903

TABLE IV

PUBLICATIONS EMPLOYING INVERSE REPRESENTATION OF ALTERNATING-CURRENT VECTORS

Name of Publication	Author	Publisher	Date
1. Dynamo-Electric Machinery	S. P. Thompson	Spon & Chamberlain	1905
2. Elements of Electrical Engi- neering	C. P. Steinmetz	El. W. & Engr.	1902
3. Electrical Energy	E. J. Berg	McGraw Pub. Co.	1908
4. Whittaker's El. Engr's. Pocket Book		Whittaker & Co.	1906
5. Electrical Engineering	Thomälin	Longmans Green	1907
6. Electrical Engineering	Rosenberg, Gee & Kinzbrunner	John Wiley	1908
7. The Induction Motor	B. A. Behrend	El. W. & Engr.	1901
8. Transformers	Gisbert Kapp	Whittaker & Co.	1903
9. Vectors and Vector Diagrams	Cramp & Smith	Longmans Green	1909
10. Alternating-Current Engineer- ing	E. B. Raymond	D. Van Nostrand	1907
11. Polyphase Currents	A. Still	Whittaker & Co.	1906
12. Practical Calculation of Trans- mission Lines	L. W. Rosenthal	McGraw Pub. Co.	1909
13. Electrical & Magnetic Calcula- tions	A. A. Atkinson	D. Van Nostrand	1903
14. Laboratory & Factory Tests in El. Engg:	Sever & Townsend	"	1907
15. Electricity and Magnetism	Foster & Atkinson	Longmans Green	1896
16. Electric Motors	H. M. Hobart	Whittaker & Co.	1904
17. Single-phase Commutator Mo- tors.	F. Punga	"	1906
18. Essais des Machines	Duquesne et Rouvière	Béranger	—
19. Stromverteilungssysteme	P. Häfner	Jänecke	1906
20. Untersuchung elektrischer Systeme.	H. Hausrath	J. Springer	1907
21. Der Drehstrommotor	J. Heubach	J. Springer	1903
22. Die Wechselstromtechnik	E. Arnold	"	1902
23. Ruhende Umformer	V. Bondi	Jänecke	1908
24. Impianti Elettrici a Correnti Alternate	A. Marro	V. Hoepli	1907

EXAMPLE OF A NON-ROTATIVE VECTOR

A simple example of a non-rotative vector is an ordinary impedance of the type z/θ ohms, as indicated in Fig. 9. An ordinary impedance does not pass through zero cyclically, like an alternating e.m.f. or current, and no practical use is at present derivable from the notion of rotating a vector impedance about its origin. In direct representation, the impedance of Fig. 9 is essentially an inductive impedance, as distinguished from a condensive impedance. That is, it represents the impedance of some particular reactance coil. With inverse representation, however, it would necessarily represent a condensive impedance, or the impedance of some particular condenser, operated at a certain frequency, in series with a certain resistance.

Impedances, admittances, reluctances and permeances, when treated as vectors, are essentially non-rotative vectors.

Quantitatively, the application of a non-rotative vector to a

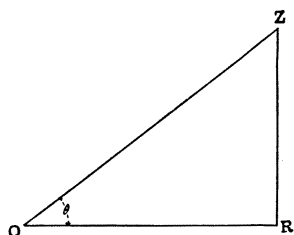


FIG. 9.—Non-rotative vector inductive impedance. Direct representation.

rotative vector as a multiplying factor alters both the magnitude and the phase of the resultant rotative vector, without altering the frequency or angular velocity of rotation. Thus the relation:

$$I \sin \omega t \times Z \sin \theta = I Z \sin (\omega t + \theta) \text{ volts } \quad (8)$$

indicates that the product of a vector current I amperes, rotating with the angular velocity ω radians per second, and a vector impedance of Z ohms, with the fixed angle θ radians, gives rise to a vector voltage $I Z$, rotating with the same angular velocity as I , but advanced θ radians in phase beyond I . The orthogonal projection of the voltage product will follow a simple harmonic motion, θ radians advanced in phase with respect to the corresponding motion of I .

DEGRADATION OF A ROTATIVE VECTOR INTO A STATIONARY VECTOR

It frequently happens that a quantity which is capable of being regarded as a rotative-vector quantity, needs only to be designated as a vector in respect of phase relation to another vector or vectors; as, for example, when a simple harmonic alternating current, of say 10 amperes r.m.s., is desired to be designated as a vector lagging, perhaps 37 degrees in phase, behind a simple

harmonic impressed e.m.f. of 100 volts r.m.s. It would be possible to give a direct representation of this condition as in Fig 2 with a rotative vector OE of 141.4 volt-scale length, followed at 37 deg. by a rotative vector OI of 14.14 ampere-scale length. But if there is no necessity for drawing attention to the orthogonally projective properties of these vectors, they may be represented as a simple non-rotative pair, OE, OI , Fig. 10, in which case it is convenient to use their virtual or root-mean-square values, instead of their maximum values. We may consider that, quantitatively, this diagram represents the following conditions:

$$OE = OI / \underline{0^\circ} \times Z / \underline{37^\circ} \quad \text{volt-scale cm. } / \underline{\quad} \quad (9)$$

and

$$OI = OE / \underline{0^\circ} \div Z / \underline{37^\circ} \quad \text{ampere-scale cm. } / \underline{\quad} \quad (10)$$

That is, there is some non-rotative vector impedance of $10 / \underline{37^\circ}$ ohms which connects, by Ohm's law, a virtual e.m.f. of 100 volts with a virtual current of 10 amperes lagging 37 deg. behind it.

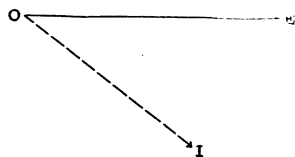


FIG. 10.—E.m.f. and lagging current. Stationary vectors.

In general, therefore, a non-rotative vector operator, such as an impedance, multiplied into a rotative vector, produces a rotative vector of changed amplitude and phase, but when multiplied either into a pure non-rotative vector, or into a stationary vector, produces a non-rotating vector of changed amplitude and phase, according to the property of multiplication of complex numbers:

$$a / \underline{\alpha} \times b / \underline{\beta} = a b / \underline{\alpha + \beta} \quad \text{numeric } / \underline{\quad} \quad (11)$$

POWER IN SIMPLE ALTERNATING-CURRENT CIRCUITS

It is well known that if a simple harmonic e.m.f. $E_0 \cos \omega t$ volts, where E_0 is the maximum cyclic value, propels a simple harmonic current $I_0 \cos (\omega t \pm \theta)$ amperes; so that the e.m.f. and current differ in phase by the positive or negative angle θ , the electric power developed in the circuit by the source of e.m.f. on the current is at any instant:

$$p_t = \frac{E_0 I_0}{2} \{ \cos \theta + \cos (2\omega t \pm \theta) \} \quad \text{watts} \quad (12)$$

$$= E I \{ \cos \theta + \cos (2\omega t \pm \theta) \} \quad \text{“} \quad (13)$$

where E and I are respectively the virtual or root-mean-square e.m.f. and current; or if P be the product $E I$, of root-mean-square volts and amperes:

$$p_t = P \{ \cos \theta + \cos (2\omega t \pm \theta) \} \quad \text{watts (14)}$$

Consequently, any projective or interceptive rotating vector which represents the power in a simple alternating-current circuit must possess an angular velocity double that of the e.m.f. or

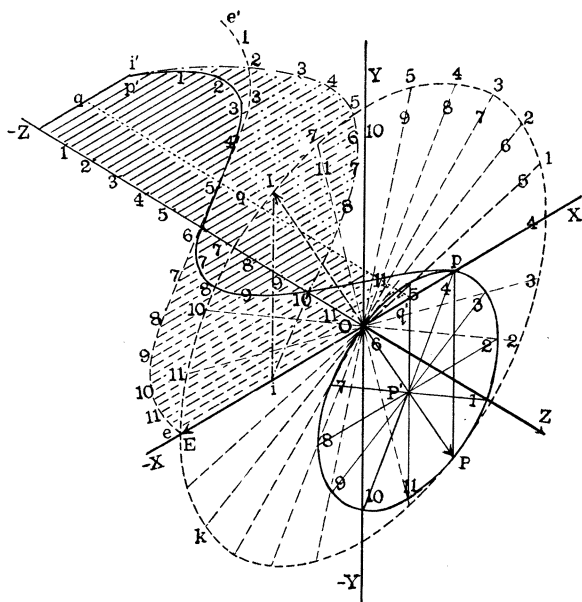


FIG. 11.—Rotative vector e.m.f. and current in inductively impendant circuit with double-frequency rotative vector power. Power factor 0.5. Isometric projection. Direct representation. XOY plane of rotation, XOZ plane of projection.

current. Moreover the origin or axis of the rotating vector power must be displaced from the origin of the e.m.f. and current components.

It has been pointed out by Mr. J. Irving Brewer¹ that a construction which satisfies the rotative-vector power relations is to lay off OE , the maximum cyclic e.m.f., along the OX axis, (Fig. 11) and OI at the proper phase angle; in the case presented,

1. "Electrical World" Jan. 21, 1909, Vol. 53, No. 4, pp. 217-218. Inverse Notation.

XOP . Then on OP at a point P' , such that $OP' = EI$, to watt-scale, is the center of the rotating power vector which starts with doubled angular velocity from the position $P'P$, at the moment when E_0 starts from OX , and I_0 from OP .

Under these conditions, the three rotating vectors will project on the fixed axis $-XOX$, and on the moving plane XOZ , the respective instantaneous values of power, e.m.f. and current.

NON-ROTATIVE VECTOR POWER

If a simple harmonic current of I root-mean-square amperes flows through an alternating-current circuit of impedance $R + jX = Z/\alpha$ ohms, (AC , Fig. 12), the root-mean-square potential-difference E volts on the circuit will be obtained by the operation:

$$E/\alpha = I/0 \times Z/\alpha = IZ/\alpha \quad \text{root-mean-square volts / } \underline{\quad} \quad (15)$$

Here I is taken either as an ordinary real number, or as a plane vector number of zero angle, and therefore at standard phase. That is, I is taken as a stationary vector, so that E is also a stationary vector e.m.f., (DF Fig. 12), advanced in phase α radians or degrees ahead of the current. The real component DE , is the effective root-mean-square component of potential difference, so far as concerns the average liberation of power from the source into the circuit, and the imaginary component $EF = jIX$, is the reactive root-mean-square component potential-difference, or the component which develops reactive power on the current. This reactive power is directed from source to circuit, and back again, in one power cycle, or in one half-cycle of current.

Again, if we multiply the non-rotative vector p.d. DF by the stationary, vector current, according to the formula:

$$P/\alpha = I/0 \times E/\alpha = IE/\alpha = EI/\alpha \quad \text{watts/} \underline{\quad} \quad (16)$$

we obtain a stationary vector power GK , advanced α degrees or radians ahead of the current. This power is the apparent power, commonly called volt-amperes. It is perhaps practically advantageous to call the unit of apparent power the volt-ampere in order to distinguish apparent power from effective power in engineering; but a volt-ampere is essentially a watt, and the apparent power is correctly stated as apparent or resultant watts,

the vector sum of effective and reactive watts¹. The effective component GH is the average power delivered to the circuit by the generator, and is usually called the "real power". The reactive power HK is, however, when considered from within the circuit, just as real as the effective power GH ; so that the term "real power" is unsuitable. The reactive power HK is the maximum cyclic power expended in transmitting energy into and out of the magnetic flux linked with the circuit, being alternately plus and minus, or from and to the generator, in successive quarter cycles of current. This energy is kept in the circuit; whereas the effective power HG transmits energy out of the circuit. The maximum reactive cyclic power HK is all internal. The effective power GH is the average of that delivered externally, is the cyclic average of the instantaneous total internal power, and is also the maximum cyclic value of the externally delivered power.

Finally, if we divide the stationary vector power by 2ω according to the equation:

$$W/\alpha = P/\alpha / (2\omega) \quad \text{joules per energy cycle / } \underline{\quad} \quad (17)$$

we obtain the stationary energy vector LN . This is the maximum cyclic apparent, or resultant oscillatory, energy delivered by the generator to the circuit in each energy cycle, over and above the average effective energy delivered at the rate GH . The perpendicular component MN is the maximum cyclic change of reactive energy in the magnetic flux of the circuit. The horizontal component LM is the maximum cyclic oscillation of effective energy.

A practical example will illustrate the above conditions. We may assume a large single-phase 60-cycle alternator delivering at its switchboard eight megawatts (8,000 kw.) of effective power, and three megawatts of reactive power, or 8.544 apparent megawatts, under a power factor of 0.936. If the delivered current is 4,000 amperes, its terminal voltage will be 2,136 volts.

The stationary-vector power diagram for this generator is shown at GHK in Fig. 12. The analysis of the circuit conditions is given in Fig. 13, to rectangular coördinates and to

1. A useful diagram of this type—essentially a stationary-vector power diagram—appears at page 540 of Mr. Percy H. Thomas' paper on "Calculation of the High-Tension Line". PROCEEDINGS of A.I.E.E., June, 1909.

sinusoidal current phase. The current I has a maximum cyclic strength of $4000\sqrt{2}$, or 5,656 amperes. The potential difference E is ahead of the current by $\alpha = 20.6$ deg. It is analyzed into the effective component (DE Fig. 12) of 2,000 volts root-mean-square, or $E_f = 2,828$ volts maximum (Fig. 13)

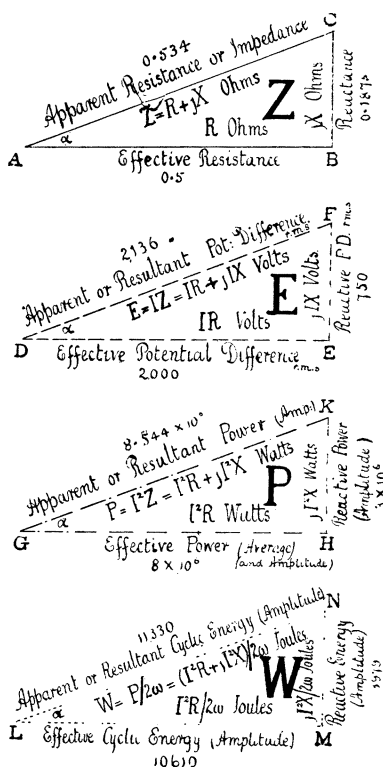


FIG. 12.—Stationary vector diagrams of impedance, potential-difference, power and cyclic energy in an alternating-current circuit to phase of current as standard.

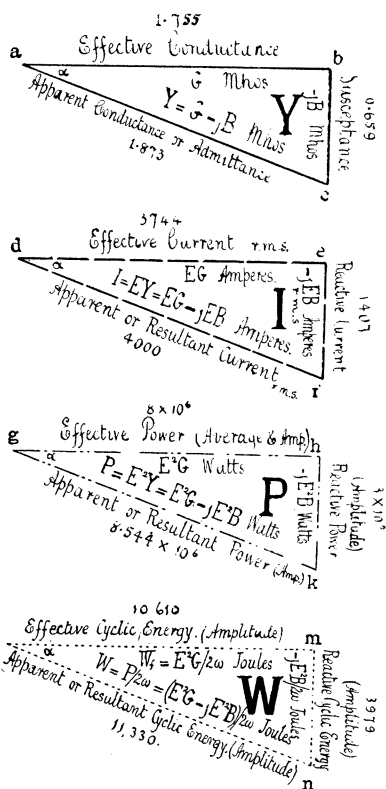


FIG. 14.—Stationary vector diagrams of admittance, current, power and cyclic energy in an alternating-current circuit to phase of potential-difference as standard.

and the reactive component EF of 750 volts root-mean-square ($E_k = 1061$ volts maximum). The product $I E_f$ produces the effective power P_f of eight megawatts amplitude, (GH Fig. 12) above and below the average P_m of eight megawatts. The product $I E_k$ produces the reactive power P_k of three megawatts

amplitude (*H K* Fig. 12). The sum of these two quadrature power components is the total apparent power $I E = P$ of 8.544 megawatts above and below the mean P_m .

The reactive power P_k is associated with a cyclic energy change of $W_k = 0.3979$ myriajoule = 3,979 joules = 405.6 kilogram-meters or 2940 ft-lb. This energy (*M N* Fig. 15) is stored

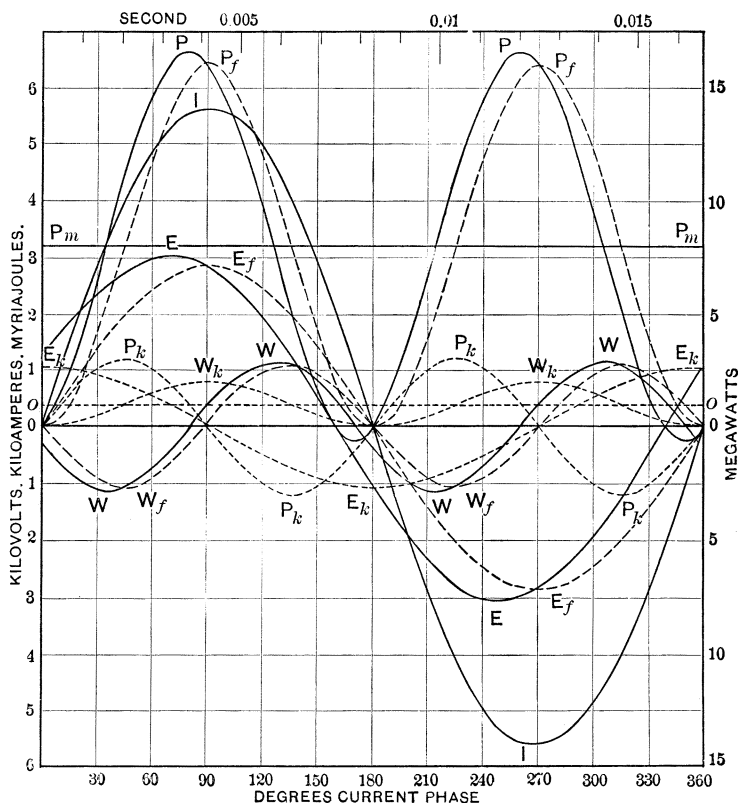


FIG. 13.—Analysis of potential-difference, power and energy to current as standard of phase.

in, or removed from, the magnetic flux of the circuit in one quarter of an energy-cycle, or one eighth of a current-cycle (0.00208 second) over and above a steady stock of energy *ooo* (Fig. 13) of equal amount. The total magnetic energy in a current half-cycle is therefore:

$$2 W_k = I^2 X / \omega \quad \text{joules per cycle of current} \quad (18)$$

or 7958 joules (0.7958 myriajoule), at 90 deg. and at 270 deg. of current-phase.

The effective power P_f is associated with a cyclic energy change of $W_f = 1.061$ myriajoule in each cycle ($L M$ Fig. 12) above and below the average of eight megajoules per second, or 13.333 myriajoules per current cycle, delivered by the generator outside of the circuit.

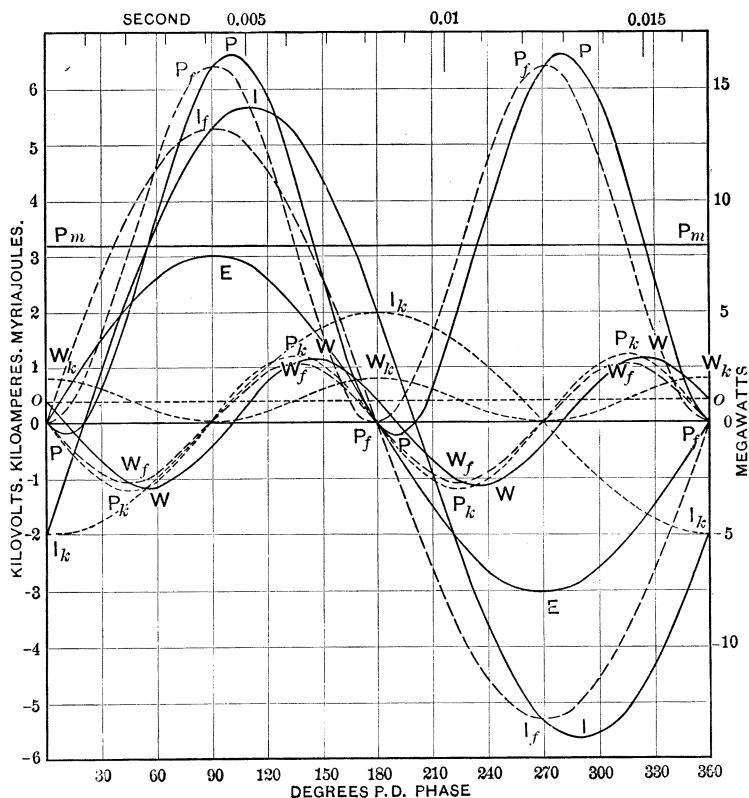


FIG. 15.—Analysis of current, power and energy to potential-difference as standard of phase.

The total cyclic energy change is $W = 1.133$ myriajoule ($L N$ Fig. 12) above and below the average of 6.667 myriajoules per energy cycle.

Fig. 12 contains, therefore, a non-rotative impedance triangle $A B C$ of ohms, a stationary potential-difference triangle $D E F$ of root-mean-square volts, a stationary power triangle $G H K$ of

maximum cyclic or amplitude watts, and a stationary energy triangle $L M N$ of maximum cyclic or amplitude joules. These four non-rotating vector triangles pertain to every alternating-current circuit, considered with reference to the stationary vector current $I/\sqrt{2}$ root-mean-square amperes. If the circuit contains condensance, instead of inductive reactance, the four triangles will all be inverted, with negative reactive quantities, or of the geometrical type indicated in Fig. 14. If the current and p. d. in the circuit are sinusoidal, then all of the 18 vector quantities involved will be strictly interpretable by simple harmonic theory. If the current, or the potential-difference, or both, are only approximately sinusoidal, the vector triangles may still be computed, and they permit of being considered as "equivalent sinusoidal" triangles.¹ In this case, however, the various vector quantities cease to be strictly interpretable physically. Finally, if the current or potential-difference, or both, depart widely from the sinusoidal type, the vector triangles, although still existing logically and geometrically, may fail completely to be interpreted physically. That is, the vector impedance, voltage, power and energy may be inconsistent with the physical conditions. In such cases, it is necessary to analyze the current and potential-difference into harmonic components, develop a series of vector triangles, one for each component, and aggregate the separate effects.²

If in a single-frequency (simple harmonic) circuit, the impedance triangle be given and the root-mean-square current, the other three vector triangles of E , P and W , follow immediately, and without any ambiguity. That is, the stationary-vector series Z , E , P and W is unique. But if either the power or energy triangle be given initially, the Z triangle which follows therefrom is ambiguous, because there may be condensance associated with the reactance, either in series or parallel, and a doubly infinite system of circuits could therefore be devised that would satisfy the W , P , and E vectors. The only definite conclusion in such a case is that the reactance preponderates over the condensance to the amount indicated by the X of the Z triangle.

If one or more impressed counter-electromotive forces exists in

1. Steinmetz, "The Law of Hysteresis", TRANSACTIONS A.I.E.E., May, 1894, Vol. 11, pp. 570, 616.

2. Steinmetz, "Symbolic Representation of General Alternating Waves and of Double-Frequency Vector Products", TRANSACTIONS A.I.E.E., June, 1899. Vol. 16, pp. 269-304.

the circuit, as, for example, that of a synchronous alternating-current motor, the Z diagram that follows from a given P diagram and current I , is logically consistent, and may be practically useful, but is purely fictitious.

If instead of taking the phase of the current as standard, we take the phase of the potential-difference as standard, we exchange $E/0$ for $I/0$ as the fundamental stationary vector, and we obtain the four stationary vectors Y, I, P , and W of Fig. 14, connected by the relations:

$$I \searrow \alpha = E/0. \quad Y \searrow \alpha \quad \text{root-mean-square amperes } \searrow \quad (19)$$

$$P \searrow \alpha = E/0. \quad I \searrow \alpha \quad \text{max. cyclic watts } \searrow \quad (20)$$

$$W \searrow \alpha = (P \searrow \alpha)/(2 \omega) \quad \text{max. cyclic joules } \searrow \quad (21)$$

It will be observed that the P and W triangles in Fig. 14 are inverted by comparison with those in Fig. 12, and yet the same single-phase alternator is supposed to be operating on the same circuit in each case. The anomaly is explained by the fact that as shown by Figs. 13 and 15, the resultant power curve P lies intermediate in phase between the current curve I and the potential-difference curve E , so that while the power is leading with respect to the current, it is lagging with respect to the potential-difference. Consequently, the stationary power vector of an alternating-current circuit has, with direct representation, either a negative or a positive angle, according as the phase of one or other of the two quantities E and I is taken as standard, as well as whether the circuit is reactive or condensive.

By comparing Figs. 13 and 15, it will be seen that the curves E, I, P and W , or resultant potential-difference, current, power and energy, all correspond, or may be superposed each on each; but the components $E_f, E_k, I_f, I_k, P_f, P_k$ and W_f, W_k lie on reversed sides of their respective resultants in the two figures, as called for by the two sets of stationary-vector diagrams in Figs. 12 and 14.

It follows therefore that the P and W stationary-vector diagrams exist in two mutually inverted forms for every single-frequency alternating-current circuit; whereas the E, I, Z and Y stationary-vector diagrams are single, their position (erect or inverted), depending upon whether the circuit is inductively or

condensively reactant, as well as upon whether direct or inverse representation is employed.

ROTATIVE PROPERTIES OF THE STATIONARY VECTORS E , I , P , AND W

In Fig. 12 the vector $E = IZ$ is taken as stationary; but it is of course capable also of being considered as a rotative vector, rotating about the vertex D , with the angular velocity ω . In that case, the three voltages E , E_f and E_k should each be increased in the ratio of $\sqrt{2}$, or should be changed in scale, in order

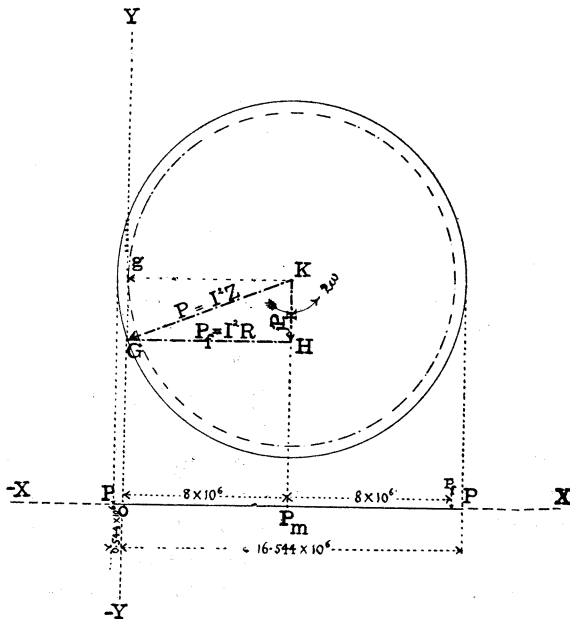


FIG. 16.—Rotative vector power diagram for current standard phase.

that their orthogonal projections on a stationary reference axis, or interceptions on a stationary circle, should correspond to the several instantaneous voltages of impressed potential-difference, effective potential-difference and reactive potential-difference.

In Fig. 12 the power vector $P = I^2 Z$ is also taken as stationary; but it is capable of being considered as a rotative vector without any change of scale, by rotating the triangle $G H K$, about the vertex K , with the positive angular velocity 2ω . This is represented in Fig. 16, where K is the center of rotation

in the plane $X O Y$ of the paper, $K g = H G$, and the extreme left-hand projection of g is taken at o , as the origin of power coördinates, in the plane $X O Z$ of projection. The projection of K , at P_m , then marks off the steady average power of the system in watts, while the projection points $P P$, marking the limits of the rotating vector $I^2 Z$, indicate the range of cyclic oscillation of the power. Thus, as shown in Fig. 13, the power pulsates between -0.544 and $+16.544$ megawatts once in each power-cycle, or half current-cycle. Similarly, the rotating vector $K H$ projects, with respect to the center P_m , the instantaneous magnitude of the reactive power P_k about the zero line $O O$, Fig. 13. In accordance with Fig. 13, the rotation of Fig. 16 is assumed to start at a moment when the vector current I , rotating with angular velocity, ω , if applied with its center on K , would occupy the direction $K H$.

Similarly, if we rotate the power triangle in Fig. 14 about its vertex k , with the angular velocity 2ω , we shall obtain the same diagram as Fig. 16, except that the phase order of advance will be reversed; *i.e.*, P_f , followed by P and P_k , instead of P_k followed by P , and P_f . The orthogonal projections of these three vectors will then correspond to the curves P_f , P and P_k as given in Fig. 15 to potential-difference phase.

In Fig. 12, the energy vector $W = I^2 Z/(2\omega)$ is also taken as stationary; but it is capable of being considered as a rotative vector without any change of scale, by rotating the triangle $L M N$ about the vertex N with the positive angular velocity 2ω , and taking instantaneous projections—not on the $X X$ axis—but on the $Y Y$ axis, as shown in Fig. 17. Here the successive rotating vectors, W_k , W , and W_f project the sinusoidal curves W_k , W , and W_f of Fig. 13, each with respect to its own zero line. At the moment represented in Fig. 17, the current I is supposed to start with angular velocity ω from the position $N M$, with center N , and to project orthogonally on the $X X$ axis.

Similarly, if we rotate the stationary energy triangle $l m n$ of Fig. 14, about the vertex n , with positive angular velocity 2ω , and project upon the $Y Y$ axis, we shall obtain a diagram like that of Fig. 17, except that the order of vector succession will be reversed, in accordance with the projected curves of Fig. 15.

But the rotating energy vector of Fig. 17 only projects the fluctuation of energy in the circuit. There is, in addition to this fluctuation of energy, a steady stream of energy delivered from the circuit, corresponding to the steady average power

tracing point then lies on the tread of the wheel. As shown in Fig. 20, the tracing point describes a cycloid in the $X O Y$ plane, and the recessional loops in the tracing path disappear. The energy delivered by the generator to the circuit then halts at each cycle of energy vector rotation, or turn of the wheel; but it does not reverse.

If, on the contrary, the impedance factor Z/R of the circuit increases, either inductively or condensively, the flange radius bears the same proportion to the tread radius. The oblate

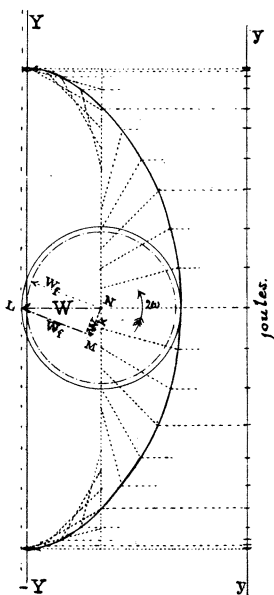


FIG. 19.—Projection energy of rotative- and rolling-vector energy triangle.

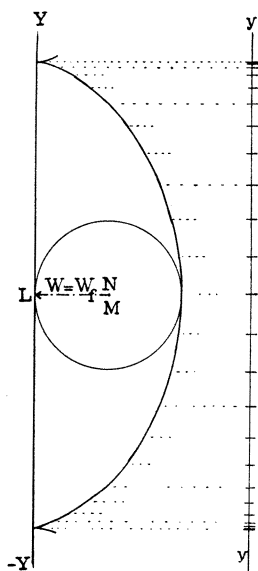


FIG. 20.—Rotating- and rolling-vector energy for non-inductive circuit.

trochoid described, as in Fig. 21, by the tracing point on the flange, develops larger recessional loops to the left of the line $Y Y$, with more marked reversals of motion and energy in the projected point.

Analytically, if W be the flange radius, and W_f the tread radius of the rolling wheel, the distance of the projected point along $y y$ from the initial position at which W lies parallel to the $-X X$ axis is:

$$w_t = W_f 2\omega t + W \sin 2\omega t \quad \text{joule-scale cm.} \quad (22)$$

or substituting for W_f and W their values $P_f/2\omega$ and $P/2\omega$ respectively,

$$w_t = P_f t + \frac{P}{2\omega} \sin 2\omega t \quad \text{joules} \quad (23)$$

$$\therefore p_t = \frac{dw_t}{dt} = P_f + P \cos 2\omega t \quad \text{watts} \quad (24)$$

as already deduced from Figs. 13 and 15.

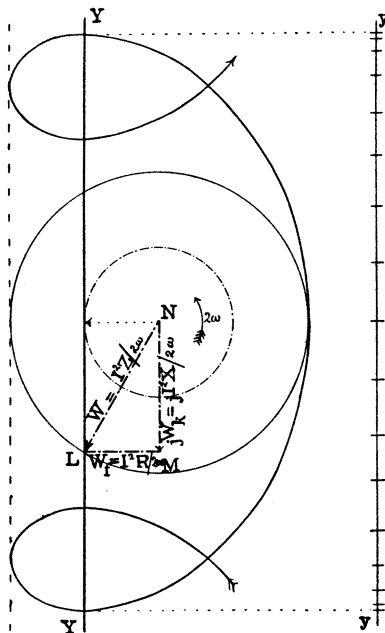


FIG. 21.—Rotative- and rolling-vector energy triangle for circuit of power factor 0.5.

Finally, if the circuit be purely reactive, or resistanceless, the tread radius becomes zero, and the wheel spins without rolling. The path of the tracing point on the flange is then a pure circle. Its projection on the YY axis is a simple harmonic motion, and on the YOZ plane, a pure sinusoid.

Consequently, with suitable convention as to direction and sign of translatory motion, the energy given to an alternating current circuit may be represented by the vertically falling shadow, on the horizontal supporting rail, of a point on the flange of a uniformly rolling wheel. The rate of motion of this shadow will be

the power given to the circuit. The average power will be represented by the uniform speed of the axle. The instantaneous power by the instantaneous velocity of the shadow. If a second wheel be geared with the axle so as to rotate in the same direction with half the angular velocity, suitably selected radii on the latter wheel will represent the p. d. and current in the circuit, the instantaneous projections of these radii being measured on a vertical coördinate axis, and not on the horizontal rail.

CONFORMITY OF THE ALGEBRA OF THE ALTERNATING-CIRCUIT AND CONTINUOUS-CURRENT CIRCUITS

Finally, it should be pointed out that just as, in regard to impedances, admittances, currents, and e.m.fs., the algebra of the single-frequency alternating-current circuit is the same as the algebra of the continuous-current circuit, the former dealing with complex quantities while the latter deals with real quantities;¹ so the algebra of both circuits is the same, or at least may be regarded as the same, in dealing with power. For the power product of an e.m.f. E/θ and a current I/θ , or of an E/θ and a current I/θ , is $E I/\theta = I E/\theta$ watts and not $E I \cos \theta$ watts. The average externally delivered power, as well as the average instantaneous power, is however $E I \cos \theta$ watts and the maximum cyclic internally-reactive power is $E I \sin \theta$ watts.

SUMMARY OF CONCLUSIONS

The algebra and geometry of vector alternating-current technology, as developed in text books, are, at present, in a state of great and unnecessary confusion as to direction of rotation.

The confusion has existed for more than twenty years, and is not confined to any one country or language.

Calling that representation "direct" which denotes a *leading* current as *leading*, in the order of positive rotation, some two thirds of the alternating-current text-books use direct representation, and the remaining one third inverse representation.

The existing dissension relates to conventions and not to facts.

The directions of rotation and representation should be standardized by mutual international agreement.

1. This law was first announced by the writer, restricted however to impedances and admittances, in the paper on "Impedance", TRANSACTIONS A.I.E.E., April 1893, Vol. 10, pp. 175-232. The law was speedily extended by Dr. C. P. Steinmetz to cover currents and e.m.fs.

The terms "real power" and "wattless power" are inaccurate and misleading.

It is readily possible to compute and discuss the power in an alternating-current circuit as a stationary-vector quantity, without reference to the double frequency of rotative-vector power.

The power developed by any single-frequency alternating e.m.f. E root-mean-square volts on a co-frequent current $I/\pm\theta$ root-mean-square amperes, is algebraically $E I/\pm\theta$ watts. The externally liberated power is $E I \cos \theta$ watts, and is usually the principal consideration in power-transmission systems.

In any alternating-current circuit, or portion of the same, there are four non-rotating vectors Z, E, P, W to standard current phase, and also four Y, I, P, W , to standard potential-difference phase, all connected by ordinary vector arithmetic, and not involving double-frequency products.

The energy in a single-frequency alternating-current circuit follows the projection, upon the rail, of a flange-point on a wheel rolling along the rail with uniform angular velocity. The path of the flange-point is an oblate trochoid for reactive circuits—but is a cycloid for a non-reactive circuit.

The algebra of alternating currents may be regarded as the same as the algebra of continuous currents, for power as well as for other quantities; so that any formula relating to direct-current circuits is also a formula relating to single-frequency alternating-current circuits, when complex numbers are substituted for real numbers.

NOTATION EMPLOYED

a, b , Vector lengths (cm.).

α, β, θ , Vector angles, or phase angles (radians or degrees).

B Susceptance of a circuit, as a whole, or beyond a pair of points in the same (mhos).

c Capacity of a condenser (farads).

$E_0, E E_f E_k$, Maximum cyclic, virtual or root-mean-square, effective, and reactive e.m.f. in a circuit, or potential-difference at a pair of points in the same (volts).

e_t Instantaneous voltage or P. D. in a circuit at time t (volts).

$\epsilon = 2.71828 \dots$ (numeric).

G Conductance of a circuit as a whole, or beyond a pair of points in the same (mhos).

- I_0, I, I_f, I_k Maximum cyclic, virtual or root-mean-square, effective, and reactive current strength in a circuit (amperes).
 i_t Instantaneous current strength in a circuit (amperes).
 $j = \sqrt{-1}$ (quadrantal operator).
 L Inductance in a circuit (henrys).
 l Polar radius-vector of a complex number.
 n Frequency of alternation (cycles per second).
 ω Angular velocity of a rotating vector, or of an alternating quantity, (radians per second, or degrees per second).
 P, P_f, P_k Resultant, effective, and reactive maximum cyclic power in a circuit (watts).
 P_m Average power in a circuit (watts).
 p_t Instantaneous power in a circuit at time t (watts).
 $\pi = 3.14159 \dots$ (numeric).
 R, r , Resistance in a circuit, or conductor (ohms).
 T , A constant time interval (seconds).
 t Elapsed time interval (seconds).
 W, W_f, W_k Resultant, effective and reactive maximum cyclic energy (joules).
 w_t Instantaneous energy in a circuit (joules).
 X, x , Reactance in a circuit (ohms).
 Y , Admittance, or resultant conductance, in a circuit (mhos).
 Z, z , Impedance, or resultant resistance in a circuit (ohms).
 $X X, Y Y, Z Z$, Rectangular coordinate axes.
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